The Idea and Structures of Geometrodynamics

Domenico Giulini

MPI für Gravitationsphysik
Potsdam

Leipzig, September 21st 2008

William Kingdon Clifford 1870

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena.
I hold in fact:

1. That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
2. That this property of being curved or distorted is continually being passed from one portion of space to another after the manner of a wave.
3. That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.
4. **That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity."**

"People slowly accustomed themselves to the idea that the physical states of space itself were the final physical reality."

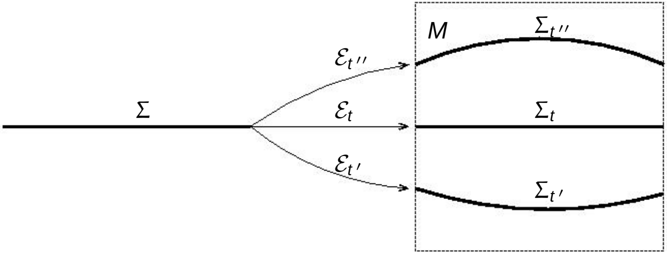
—PROFESSOR ALBERT EINSTEIN

The New Yorker

Einstein's equation

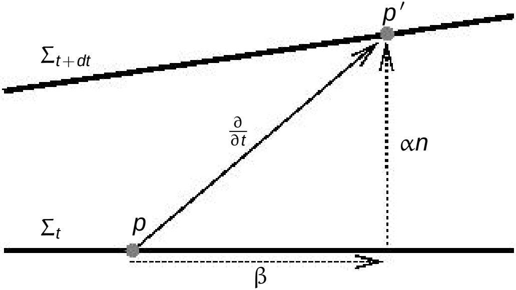
$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

Spacetime as space's history



Spacetime, M, is foliated by a one-parameter family of embeddings $\mathcal{E}_t$ of the 3-manifold Σ into M. Σ_t is the image in M of Σ under $\mathcal{E}_t$.

A four-function worth of arbitrariness



For $q \in \Sigma$ the image points $p = \mathcal{E}_t(q)$ and $p' = \mathcal{E}_{t+dt}(q)$ are connected by the vector $\partial/\partial t|_p$ whose components tangential and normal to Σ_t are β (three functions) and αn (one function) respectively.

Kinematics of hypersurface deformations

- In local coordinates y^μ of M and x^m of Σ the generators of normal and tangential deformations of the embedded hypersurface are

$$
\begin{aligned}
N_\alpha &= \int_\Sigma d^3x\ \alpha(x)\, n^\mu[y(x)]\, \frac{\delta}{\delta y^\mu(x)} \\
T_\beta &= \int_\Sigma d^3x\ \beta^m(x)\, \partial_m y^\mu(x)\, \frac{\delta}{\delta y^\mu(x)}
\end{aligned}
$$

- This is merely the foliation-dependent decomposition of the tangent vector $X(V)$ at $y \in \mathrm{Emb}(\Sigma, M)$, induced by the spacetime vector field $V = \alpha n + \beta^a \partial_a$:

$$X(V) = \int_\Sigma d^3x\ V^\mu(y(x))\, \frac{\delta}{\delta y^\mu(x)}$$

- The vector fields $X(V)$ on $\mathrm{Emb}(\Sigma, M)$ obey

$$[X(V), X(W)] = X\big([V, W]\big)\,,$$

i.e. $V \mapsto X(V)$ is a Lie homomorphism from the tangent-vector fields on M to the tangent-vector fields on $\mathrm{Emb}(\Sigma, M)$.

- In terms of the normal-tangential decomposition:

$$\begin{aligned}
[T_\beta\,,\,T_{\beta'}] &= -T_{[\beta,\beta']}\,,\\
[T_\beta\,,\,N_\alpha] &= -N_{\beta(\alpha)}\,,\\
[N_\alpha\,,\,N_{\alpha'}] &= -\epsilon\,T_{\alpha\,\mathrm{grad}_h(\alpha')-\alpha'\mathrm{grad}_h(\alpha)}\,,
\end{aligned}$$

- Here $\epsilon = 1$ for Lorentzian and $= -1$ for Euclidean spacetimes, just to keep track of signature dependence.

Hamiltonian GR

- The idea is to represent the algebraic structure of hypersurface deformations in terms of a Hamiltonian dynamical system of physical fields.
- **Theorem:** The most general local realisation on the cotangent bundle over $\mathrm{Riem}(\Sigma)$, coordinatised by (h, π), is

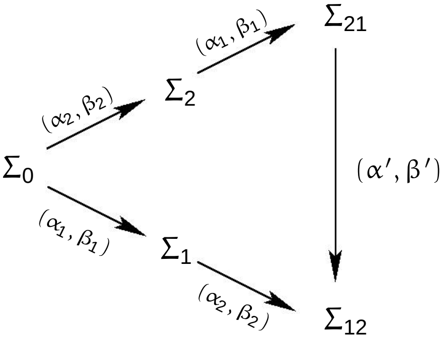
$$N_\alpha \mapsto H_\alpha[h,\pi] := \int_\Sigma \alpha(x)\,\mathcal{H}[h,\pi](x)$$

$$T_\beta \mapsto D_\beta[h,\pi] := \int_\Sigma \beta^a(x)\,h_{ab}(x)\,\mathcal{D}^b[h,\pi](x)$$

where

$$\mathcal{H}[h,\pi] := \epsilon(2\kappa)G_{ab\,cd}\pi^{ab}\pi^{cd} - (2\kappa)^{-1}\sqrt{h}\,(R-2\Lambda)$$

$$\mathcal{D}^b[h,\pi] := -2\,\nabla_a\pi^{ab}$$



Successive hypersurface deformations parametrised by (α_1, β_1) and $N_2 = (\alpha_2, \beta_2)$ do not commute; rather

$$[X(\alpha_1, \beta_1), X(\alpha_2, \beta_2)] = X(\alpha', \beta'),$$

where

$$\begin{aligned} \alpha' &= \beta_1(\alpha_2) - \beta_2(\alpha_1), \\ \beta' &= [\beta_1, \beta_2] + \alpha_1 \operatorname{grad}_h(\alpha_2) - \alpha_2 \operatorname{grad}_h(\alpha_1). \end{aligned}$$

- Since α' depends on h, we get the following condition for the Hamiltonians to act (via Poisson Bracket) as derivations on phase-space functions:

$$
\begin{aligned}
&\big\{\{F,H(\alpha_1,\beta_1)\},H(\alpha_2,\beta_2)\big\}-\big\{\{F,H(\alpha_2,\beta_2)\},H(\alpha_1,\beta_1)\big\}\\
&=\ \big\{F,\{H(\alpha_1,\beta_1),H(\alpha_2,\beta_2)\}\big\}\ =\ \big\{F,H(\alpha',\beta')\big\}\\
&=\ \{F,H\}(\alpha',\beta')+H\big(\{F,\alpha'\},\{F,\beta'\}\big)\\
&\overset{!}{=}\ \{F,H\}(\alpha',\beta')
\end{aligned}
$$

- The last equality must hold for all F and all (α_1,β_1) (α_2,β_2). This implies the constraints:

$$\mathcal{H}[h,\pi](x)=0 \qquad \mathcal{D}^a[h,\pi](x)=0$$

- Constraints correspond to $\perp\perp$ and $\perp\parallel$ components of Einstein's equation. A spacetime in which constraints are satisfied for *each* Σ must obey Einstein's equation.
- The constraints do not cause topological obstructions to Cauchy surface. Only special requirements do, like e.g. time-symmetry.

Connection Variables

- The phase space of GR may be described by a (complex) $SO(3)$ connection A^i_a and a densitised 3-bein $\tilde{E}^a_i$, where

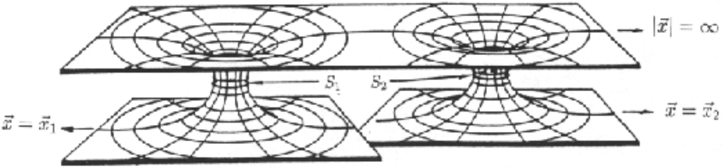
$$A^i_a := \Gamma^i_a + \beta K^i_a\,, \quad \beta = \text{Immirzi parameter}\,.$$

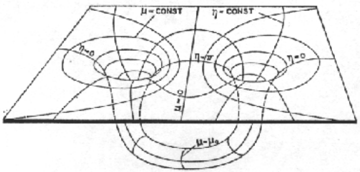
- The Hamiltonian constraint reads:

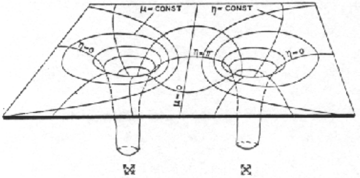
$$\begin{aligned}&\varepsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{ab\,k}\\ &-2(1+\beta^{-2})\,\tilde{E}^a_{[i}\tilde{E}^b_{j]}\,(A^i_a-\Gamma^i_a)(A^j_b-\Gamma^j_b)=0\,.\end{aligned}$$

- Unless $\beta = i$ the connection A^i_a cannot be thought of as restriction to space of a spacetime connection. For example, its holonomy along a spacelike curve γ in spacetime depends on the choice of $\Sigma \supset \gamma$.

Topologies for two BHs







Mass without mass

- The mass-energy of an asymptotically flat end is

$$m \propto \lim_{R\to\infty} \left\{ \int_{S^2_R \subset \Sigma} d\sigma (\partial_a h_{ab} - \partial_b h_{aa}) n^b \right\}$$

- This is ≥ 0 and $= 0$ for Minkowski slices only.
- Gannon's theorem implies causal geodesic incompleteness if $\pi_1(\Sigma) \neq 1$ (replacing $\exists$ trapped surfaces in the hypotheses).
- Stationary regular vacuum solutions (gravitational solitons) do not exist (Einstein & Pauli, Lichnerowicz).

Momenta without momenta

- The linear and angular momenta of an asymptotically flat end is

$$p^a \propto \int_{S^2_\infty} d\sigma\, \pi^{ab} n_b\,, \qquad J^a \propto \int_{S^2_\infty} d\sigma\, \varepsilon_{abc}\, x^b \pi^{cd} n_d$$

- Axisymmetric vacuum configurations with $J \neq 0$ and one end do not exist, even for non-orientable Σ:

$$J_K = \int_{S^2_\infty} \star dK = \int_\Sigma \underbrace{d \star dK}_{\propto \mathrm{Ric}} = 0$$

- But for Killing fields K up to sign they do (Friedman & Mayer 1981).

Charge without charge

- Electrovac solutions with non-zero overall electric charge

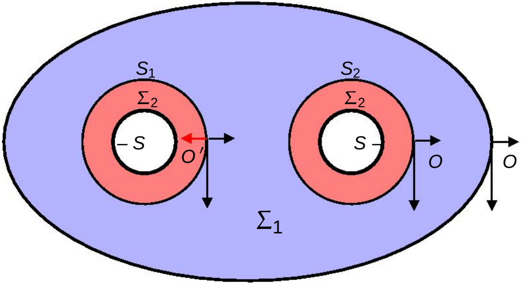
$$Q_e = \int_{S^2_\infty} \star F$$

only exist if $S^2_\infty \neq \partial\Sigma$, i.e. if $[S^2_\infty] \in H^2(\Sigma)$ is non-trivial, like e.g. in Reissner-Nordström.

- If Σ has only one end and is *non-orientable*, Stokes' theorem obstructs existence of electric but not of magnetic charge (Sorkin 1977):

$$Q_m = \int_{S^2_\infty} F$$

- This is because for non-orientable Σ, Stokes' theorem holds for twisted (densitised) but not for ordinary forms.



- Stokes' theorem applied to $\vec{\nabla}\cdot\vec{B}=0$ in Σ_1:

$$\Phi(\vec{B},\partial\Sigma_1,O)+\Phi(\vec{B},S_1,O)+\Phi(\vec{B},S_2,O)=0$$

- Stokes' theorem applied to $\vec{\nabla}\cdot\vec{B}=0$ in Σ_2:

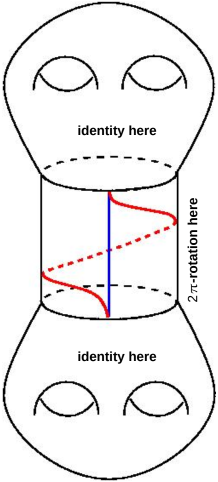
$$\Phi(\vec{B},S_1,O')+\Phi(\vec{B},S_2,O)=0$$

- Hence

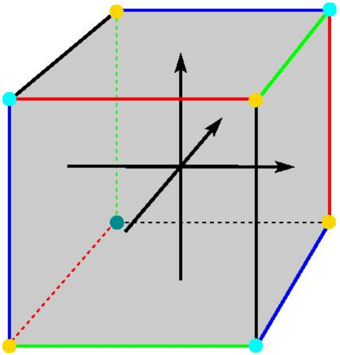
$$\Phi(\vec{B},\partial\Sigma_1,O)=-2\,\Phi(\vec{B},S_1,O)\neq 0$$

Spin without spin

- There exist many 3-manifolds for which a full (i.e. 2π) relative rotation is not in the id-component.
- In this case the asymptotic symmetry group at spacelike infinity contains $SU(2)$ rather than $SO(3)$.
- This has been suggested as a ‘fermions-from-bosons’ mechanism in gravity (Friedman & Sorkin 1982).
- The spinoriality-status of each known 3-manifold is also known.



Example: The space form S^3/D_8^*



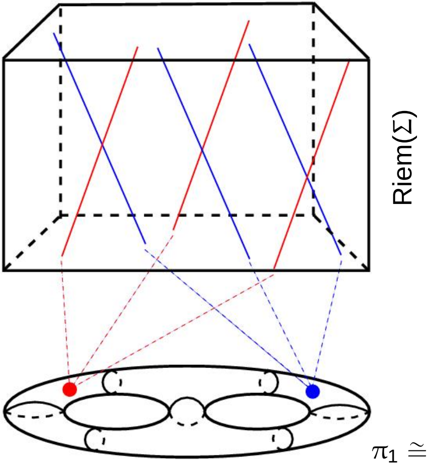
- $\Sigma = S^3/D_8^*$ is spinorial
- $D_8^* = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$
- $\mathrm{MCG}_\infty(\Sigma) \cong \mathrm{Aut}(D_8^*) \cong O$
- $\mathrm{MCG}_F(\Sigma) \cong \mathrm{Aut}_{\mathbb{Z}_2}(D_8^*) \cong O^*$
- This manifold is also chiral, i.e. it admits no orientation-reversing self-diffeomorphism (like many other 3-manifolds)

Chirality of spherical space-forms

$$
\begin{tikzcd}
L(4,1) \arrow[r, dotted, "\tilde{f}"] \arrow[d, "p"'] \arrow[dr, "f\circ p"] & L(4,1) \arrow[d, "p"] \\
S^3/G \arrow[r, "f"'] & S^3/G
\end{tikzcd}
$$

$$\deg(\tilde{f}) \cdot \deg(p) = \deg(p) \cdot \deg(f)$$

Superspace



$\pi_1 \cong \mathrm{MCG}_F(\Sigma)$

Geometry

- The group $\mathrm{Diff}_F(\Sigma)$ acts as isometries on the **Wheeler-DeWitt metric** on $\mathrm{Riem}(\Sigma)$

$$\mathcal{G}_h(k,\ell) := \int_\Sigma d^3x\; G^{ab\,cd}[h](x)\; k_{ab}(x)\;\ell_{cd}(x)\,.$$

Hence G defines a metric on superspace in the usual way iff horizontal lifts are unique.

- Now, the vertical subspace at $h \in \mathrm{Riem}(\Sigma)$ is spanned by all vectors of the form

$$X^\xi = \int_\Sigma d^3x\; L_\xi h_{ab}\,\frac{\delta}{\delta h_{ab}}\,, \quad \forall \xi\,.$$

Hence $k \in T_h\mathrm{Riem}(\Sigma)$ is horizontal ($\mathcal{G}$-orthogonal to all vertical vectors) iff

$$\mathcal{O}_h k = 0 \Leftrightarrow \nabla^b(k_{ab} - \lambda h_{ab} k^c_c) = 0$$

- Horizontal projection of $k \in T_h\mathrm{Riem}(\Sigma)$: solve $\mathcal{O}_h(k + X^{\xi})$ for ξ modulo Killing fields. This is equivalent to

$$\big(\delta d + 2(1-\lambda)d\delta - 2\mathrm{Ric}\big)\,\xi = -\mathcal{O}_h k\,.$$

- Killing fields $\in$ kernel of the l.h.s. operator (symmetric) and right hand side is L^2-orthogonal to Killing fields. Non-Killing ξ in the kernel correspond precisely to those non-zero X^{ξ} that form the non-trivial intersection of the vertical and horizontal subspace of $T_h\mathrm{Riem}(\Sigma)$, which exist for $\mathcal{G}$ non-pos. def. ($\lambda > 1/3$).
- Example: $\xi = d\phi$ generate inf. dim. intersection at (locally) flat metrics in case $\lambda = 1$, whereas no non-trivial intersection exists at $\mathrm{Ric} < 0$ metrics, which always exist, and at non-flat Einstein metrics (space forms).
- The symbol of the l.h.s. operator is

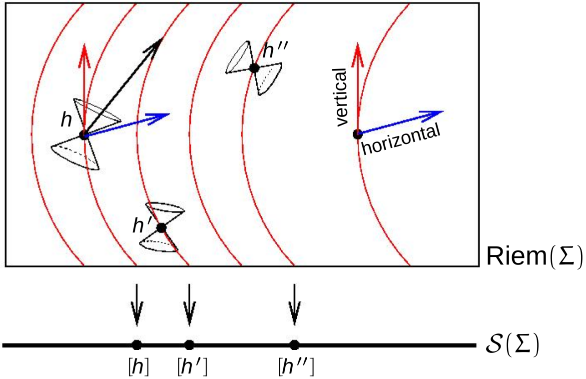
$$\sigma(\zeta)^a_b = \|\zeta\|^2\left(\delta^a_b + (1-2\lambda)\zeta^a\zeta_b/\|\zeta\|^2\right)\,,$$

which is elliptic for $\lambda \neq 1$ (strongly for $\lambda > 1$) and degenerate for the GR case $\lambda = 1$.

- Let now $\lambda = 1$ (GR value). Since then $\mathcal{G}$ defines a metric at points $[h] \in \mathcal{S}(\Sigma)$ where h is Einstein, it does so for the round metric on $\Sigma = S^3$.
- We ask: What is the signature (n_-, n_+) of the metric that $\mathcal{G}$ defines in $T_{[h]} \in \mathcal{S}(\Sigma)$? The answer is given by the following

Theorem

The Wheeler–DeWitt metric in a neighbourhood of the round 3-sphere in $\mathcal{S}(\Sigma)$ is of signature $(-1, \infty)$, that is, it is an infinite-dimensional Lorentzian metric. (DG 1995)



h
h'
h''
vertical
horizontal
Riem(Σ)
$\mathcal{S}(\Sigma)$
$[h]$
$[h']$
$[h'']$

Topology

- MCG of 3-manifolds are the π_1 of corresponding superspace. They are topological but *not* homotopy invariants.
- Consider 'lens spaces' $L(p,q) =: S^3/\sim$, where $q<p$ are coprime integers, with $S^3 = \{|z_1|^2 + |z_1|^2 = 1 \mid (z_1,z_2)\in\mathbb{C}^2\}$ and $(z_1,z_2)\sim(z_1',z_2') \Leftrightarrow z_1' = \exp(2\pi i/p)z_1$ and $z_2' = \exp(2\pi iq/p)z_2$.
- Have homotopy ($\simeq$) and topological ($\cong$) equivalence properties (Whitehead 1941, Reidemeister 1935):

$$\begin{aligned} L(p,q) &\simeq L(p,q') \Leftrightarrow q'q = \pm n^2 \pmod p \\ L(p,q) &\cong L(p,q') \Leftrightarrow q' = \pm q^{\pm 1} \pmod p \quad [q' = q^{\pm 1} \text{ for } o.p.] \end{aligned}$$

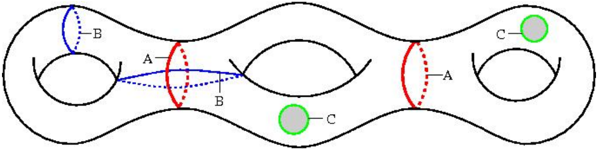
- For $p>2$ have:

$$\mathrm{MCG}_F(L(p,q)) = \begin{cases} \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } q^2 = 1 \pmod p \text{ and } q\neq\pm 1 \pmod p \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

- For example, $L(15,1)\simeq L(15,4)$ and $L(15,1)\not\cong L(15,4)$, but

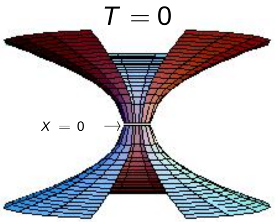
$$\mathrm{MCG}_F(L(15,1)) = \mathbb{Z}_2 \neq \mathbb{Z}_2\times\mathbb{Z}_2 = \mathrm{MCG}_F(L(15,4))$$

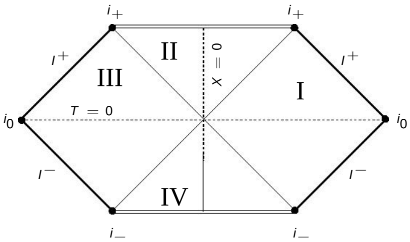
Connected sums



- Decompose along *splitting* and *essential* 2-spheres until only *prime-manifolds* remain. Prime factors are unique up to permutation.
- Except for $S^1 \times S^2$, a prime manifold has trivial π_2. The converse is true given PC. Given TGC, all finite-π_1 primes are spherical space-forms S^3/G, $G \subset SO(4)$. Infinite-π_1 primes are $S^1 \times S^2$, the flat ones $\mathbb{R}^3/G$, $G \subset E_3$, and the huge family of locally hyperbolic ones.

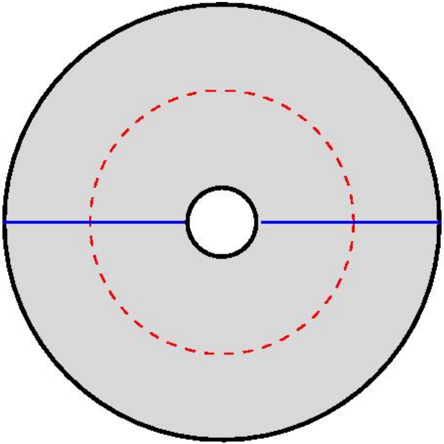
The $\mathbb{RP}^3$ geon



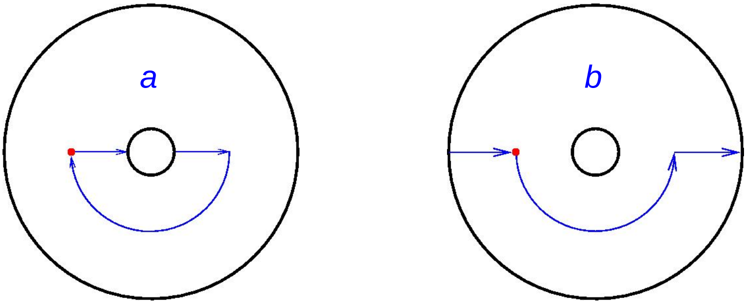


$$(T, X, \theta, \varphi) \mapsto (T, -X, \pi - \theta, \varphi + \pi)$$

$\mathbb{R}P^3\#\mathbb{R}P^3$



Its fundamental group



$$\underbrace{\langle a, b \mid a^2 = 1 = b^2 \rangle}_{\mathbb{Z}_2 * \mathbb{Z}_2} = \underbrace{\langle a, c \mid a^2 = 1,\ aca^{-1} = c^{-1} \rangle}_{\mathbb{Z}_2 \ltimes \mathbb{Z}}, \quad c := ab$$

MCG and its u.i. representations

- The group of mapping classes is given by

$$\mathrm{MCG}_F \cong \mathrm{Aut}(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle E, S \mid E^2, S^2 \rangle$$
$$E : (a,b) \to (b,a), \quad S : (a,b) \to (a, aba^{-1})$$

$\Rightarrow$ $ES + SE \subset$ centre of group algebra. Hence $\{1, E, S, ES\}$ generate algebra of irreducible representing operators.

$\Rightarrow$ Linear irreducible representations are at most 2-dimensional. They are: $E \mapsto \pm 1$, $S \mapsto \pm 1$ and, for $0 < \theta < \pi$,

$$E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$S \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

$\Rightarrow$ There are two 'statistics sectors', which get 'mixed' by S; the 'mixing angle' is θ.

Prime Π	HC	S	C	N	$H_1(\Pi)$	$\pi_0(D_F(\Pi))$	$\pi_1(D_F(\Pi))$	$\pi_k(D_F(\Pi))$
S^3/D_8^*	$+$	$+$	$+$	$-$	$Z_2 \times Z_2$	O^*	0	$\pi_k(S^3)$
S^3/D_{8n}^*	$+$	$+$	$+$	$-$	$Z_2 \times Z_2$	D_{16n}^*	0	$\pi_k(S^3)$
$S^3/D_{4(2n+1)}^*$	$+$	$+$	$+$	$+$	Z_4	$D_{8(2n+1)}^*$	0	$\pi_k(S^3)$
S^3/T^*	?	$+$	$+$	$-$	Z_3	O^*	0	$\pi_k(S^3)$
S^3/O^*	w	$+$	$+$	$+$	Z_2	O^*	0	$\pi_k(S^3)$
S^3/I^*	?	$+$	$+$	$-$	0	I^*	0	$\pi_k(S^3)$
$S^3/D_8^* \times Z_p$	$+$	$+$	$+$	$-$	$Z_2 \times Z_{2p}$	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{8n}^* \times Z_p$	$+$	$+$	$+$	$-$	$Z_2 \times Z_{2p}$	$Z_2 \times D_{16n}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{4(2n+1)}^* \times Z_p$	$+$	$+$	$+$	$+$	Z_{4p}	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T^* \times Z_p$	?	$+$	$+$	$-$	Z_{3p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/O^* \times Z_p$	w	$+$	$+$	$+$	Z_{2p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/I^* \times Z_p$	?	$+$	$+$	$-$	Z_p	$Z_2 \times I^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D'_{2^k(2n+1)} \times Z_p$	$+$	$+$	$+$	$+$	$Z_p \times Z_{2^k}$	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T'_{8\cdot 3^m} \times Z_p$	?	$+$	$+$	$-$	$Z_p \times Z_{3^m}$	O^*	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_1)$	w	$-$	$+$	$(-)^p$	Z_p	Z_2	Z	$\pi_k(S^3)$
$L(p,q_2)$	$w+$	$-$	$+$	$(-)^p$	Z_p	$Z_2 \times Z_2$	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_3)$	w	$-$	$-$	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_4)$	w	$-$	$+$	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
RP^3	$+$	$-$	$-$	$+$	Z_2	1	0	0
S^3	$+$	$-$	$-$	$-$	1	1	0	0
$S^2 \times S^1$	/	$-$	$-$	$+$	Z	$Z_2 \times Z_2$	Z	$\pi_k(S^3) \times \pi_k(S^2)$
R^3/G_1	/	$+$	$-$	$+$	$Z \times Z \times Z$	$\mathrm{St}(3,Z)$	0	$\pi_k(S^3)$
R^3/G_2	/	$+$	$-$	$+$	$Z \times Z_2 \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_2)$	0	$\pi_k(S^3)$
R^3/G_3	/	$+$	$+$	$+$	$Z \times Z_3$	$\mathrm{Aut}_+^{Z_2}(G_3)$	0	$\pi_k(S^3)$
R^3/G_4	/	$+$	$+$	$-$	$Z \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_4)$	0	$\pi_k(S^3)$
R^3/G_5	/	$+$	$+$	$+$	Z	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
R^3/G_6	/	$+$	$+$	$-$	$Z_4 \times Z_4$	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
$S^1 \times R_g$	/	$+$	$-$	$-$	$Z \times Z_{2g}$	$\mathrm{Aut}_+^{Z_2}(Z \times F_g)$	0	$\pi_k(S^3)$
$K(\pi,1)_{\mathrm{sl}}$	/	$+$	$*$	$*$	$A\pi$	$\mathrm{Aut}_+^{Z_2}(\pi)$	0	$\pi_k(S^3)$

taken from D.G. 1996